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A Non-Local Approach for SAR and Interferometric SAR Denoising

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July 27, 2010

Motivation

Why non-local methods ?

(a) Noisy image

(b) Non-local means

Noise reduction + resolution preservation

Why non-local methods ?

(a) Noisy image

(b) Non-local means

Noise reduction + resolution preservation

- Goal: to adapt non-local methods to SAR and InSAR data,
- Method: take into account the statistics of SAR and InSAR data,

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Non-local means (NL means) (1/2)

Background

- Local filters (e.g. boxcar, Gaussian filter...) $\Rightarrow$ resolution loss,
- To combine similar pixels instead of neighboring pixels.

$$\hat{u}_s = \frac{1}{Z} \sum_t e^{-\frac{|s-t|^2}{\rho^2}} v_t$$

Gaussian filter

$$\hat{u}_s = \frac{1}{Z} \sum_t e^{-\frac{\mathbf{sim}(s,t)}{h^2}} v_t$$

[Yaroslavsky, 1985]

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Non-local means [Buades et al., 2005]

- Similarity evaluated using square patches Δ_s and Δ_t centered on s and t,
- Consider the redundant structure of images.

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- Hyp. 1: similar neighborhood $\Rightarrow$ same central pixel,
- Hyp. 2: each patch is redundant (can be found many times).

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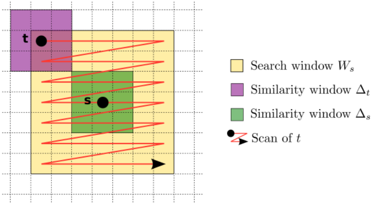


Algorithm of NL means

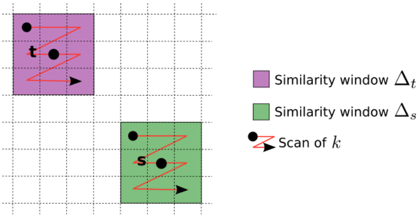
Non-local means (NL means) (2/2)

Similarity criterion [Buades et al., 2005]

- Euclidean distance:

$$\mathbf{sim}(s,t) = \sum_k |v_{s,k} - v_{t,k}|^2$$

with k the k-th respective pixel of Δ_s and Δ_t.



Euclidean distance: comparison pixel by pixel

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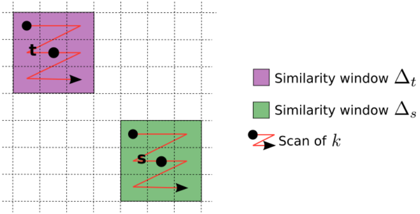

Euclidean distance: comparison pixel by pixel

- Hyp. 1: similar neighborhood $\Rightarrow$ same central pixel,
- Hyp. 2: each patch is redundant (can be found many times),
- Hyp. 3: the noise is additive, white and Gaussian.

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Weighted maximum likelihood for SAR data

Context and notations

- Searched data: R_s the reflectivity,
- Noise model: $p(.|R_s)$ a Rayleigh distribution,
- Noisy observations: A_s the amplitude such that $A_s \sim p(.|R_s)$,
- To denoise: to search an estimate $\hat{R}_s$ of R_s.

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From weighted average to weighted maximum likelihood

- NL means perform a weighted average of noisy pixel values,
- For SAR data, we suggest to use a weighted maximum likelihood:

$$\hat{R}_s = \arg\max_R \sum_t w(s,t) \log p(A_t|R) = \frac{\sum_t w(s,t) A_t^2}{\sum_t w(s,t)}$$

where $w(s,t)$ is a weight approaching the indicator function of the set of redundant pixels (i.e with i.i.d values): $\{s,t|R_s = R_t\}$.

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  where $w(s,t)$ is a weight approaching the indicator function of the set of redundant pixels (i.e with i.i.d values): $\{s, t | R_s = R_t\}$.
- Choice of $w(s,t)$: oriented, adaptive or non-local neighborhoods.

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Example (Refined Lee - oriented neighborhood [Lee, 1981])

- Redundant pixels are located in one of these eight neighborhoods:

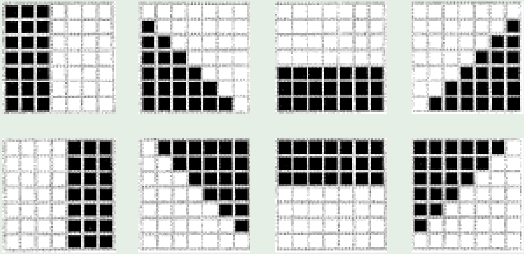

image extracted from [Lee, 1981]

Weighted maximum likelihood for SAR data

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Example (IDAN - adaptive neighborhood [Vasile et al., 2006])

- Redundant pixels are located in an adaptive neighborhood (obtained by region growing algorithm):

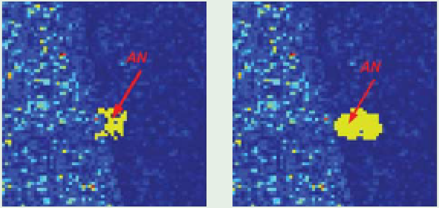

image extracted from [Vasile et al., 2006]

Context and notations

- Searched data: R_s the reflectivity,
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Example (NL means - non-local neighborhood)

- Redundant pixels are located anywhere in the image:

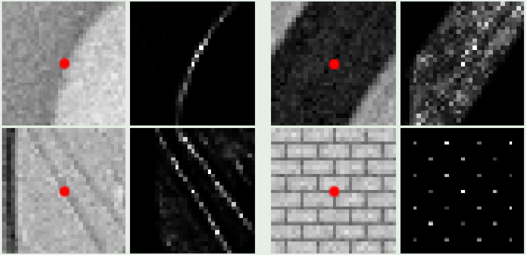

image extracted from [Buades et al., 2005]

- We search weights $w(s, t)$ such that:
  - $w(s, t)$ is high if $R_s = R_t$,
  - $w(s, t)$ is low if $R_s \neq R_t$,

Setting of the weights for SAR data (1/3)

- We search weights $w(s, t)$ such that:
  - $w(s, t)$ is high if $R_s = R_t$,
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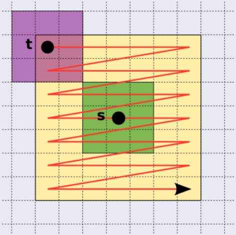
Statistical similarity between noisy patches [Deledalle et al., 2009]

- Using the hypothesis of NL means:

  Patches Δ_s and Δ_t similar $\Rightarrow$ central values s and t close.
- Weights are defined as follows:



$$w(s,t) = p(R_{\Delta_{s,k}} = R_{\Delta_{t,k}} | A_{s,k}, A_{t,k})^{1/h}$$
$$= \prod_k p(R_{s,k} = R_{t,k} | A_{s,k}, A_{t,k})^{1/h}$$

with $p(R_{s,k} = R_{t,k} | A_{s,k}, A_{t,k})$ statistical similarity

h regularization parameter.

Setting of the weights for SAR data (2/3)

Bayesian decomposition

$$p(R_1 = R_2 | A_1, A_2) \propto \underbrace{p(A_1, A_2 | R_1 = R_2)}_{\text{similarity likelihood}} \times \underbrace{p(R_1 = R_2)}_{\textit{a priori} \text{ similarity}}$$

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Similarity likelihood

$$p(A_1, A_2 | R_1 = R_2) = \frac{\int p(A_1 | R_1 = R) p(A_2 | R_2 = R) p(R_1 = R) p(R_2 = R) \, dR}{\int p(R_1 = R) p(R_2 = R) \, dR}$$

- Since $p(R_1 = R)$ is unknown, we define:

$$p(A_1, A_2 | R_1 = R_2) \triangleq \int p(A_1 | R_1 = R) p(A_2 | R_2 = R) \, \mathrm{d}R$$

- The corresponding similarity criterion is then:

$$-\log p(A_1, A_2 | R_1 = R_2) \propto \log \left(\frac{A_1}{A_2} + \frac{A_2}{A_1} \right)$$

Setting of the weights for SAR data (2/3)

Bayesian decomposition

$$p(R_1 = R_2 | A_1, A_2) \propto \underbrace{p(A_1, A_2 | R_1 = R_2)}_{\text{similarity likelihood}} \times \underbrace{\boxed{p(R_1 = R_2)}}_{\textit{a priori}\text{ similarity}}$$

Refined similarity

- Refining the weights is necessary when the signal to noise ratio is low.
- Using a pre-estimate $\hat{R}$ at pixel 1 and 2 provides two estimations of the noise models:

$$p(.|\hat{R}_1) \quad \text{and} \quad p(.|\hat{R}_2).$$

- Assuming $p(R_1 = R_2)$ depends on the proximity of $p(.|\hat{R}_1)$ to $p(.|\hat{R}_2)$, then we define

$$p(R_1 = R_2) \triangleq \exp\left(-\frac{SD_{KL}(\hat{R}_1 || \hat{R}_2)}{T}\right)$$

$$\text{with } SD_{KL}(\hat{R}_1, \hat{R}_2) \propto \frac{(\hat{R}_1 - \hat{R}_2)^2}{\hat{R}_1 \hat{R}_2}$$

- The Kullback-Leibler divergence provides a statistical test of the hypothesis $R_1 = R_2$ [Polzehl and Spokoiny, 2006]

Iterative scheme

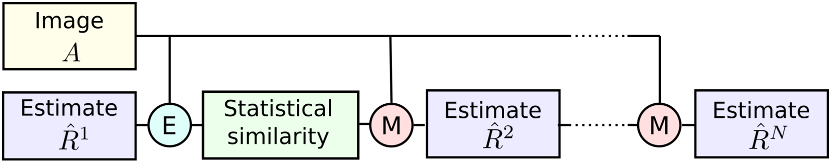
- The refined similarity involves an iterative scheme in two steps:
  1. Estimate the weights from A and $\hat{R}^{i-1}$:

$$-\log w(s,t) \leftarrow \frac{1}{h}\sum \log\left(\frac{A_1}{A_2}+\frac{A_2}{A_1}\right)+\frac{1}{T}\sum \frac{(\hat{R}_1^{i-1}-\hat{R}_2^{i-1})^2}{\hat{R}_1^{i-1}\hat{R}_2^{i-1}},$$

  2. Maximize the weighted likelihood:

$$\hat{R}_s^i \leftarrow \frac{\sum_t w(s,t)A_t^2}{\sum_t w(s,t)}$$

- The procedure converges in about ten iterations.



Scheme of the iterative filtering process

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Results of NL-SAR (Toulouse, France, TerraSAR-X ©DLR)

(a) SAR-SLC

(b) Refined Lee [Lee et al., 2003]

Results of NL-SAR (Toulouse, France, TerraSAR-X ©DLR)

(a) SAR-SLC

(b) WinSAR [Achim et al., 2003]

Results of NL-SAR (Toulouse, France, TerraSAR-X ©DLR)

(a) SAR-SLC (b) **NL-SAR**

Statistical model of InSAR data

$$p(z, z'|\Sigma) = \frac{1}{\pi^2 det(\Sigma)} \times \exp\left[-\left(z^* z'^*\right) \Sigma^{-1} \begin{pmatrix} z \\ z' \end{pmatrix} \right]$$

$$\Sigma = \mathbb{E}\left\{ \begin{pmatrix} z \\ z' \end{pmatrix} \left(z^* z'^*\right) \right\} = \begin{pmatrix} R & RDe^{j\beta} \\ RDe^{-j\beta} & R \end{pmatrix}$$

- Observations:
  - z and z' two co-registered single-look complex values.
- Parameters to estimate:
  - R the reflectivity,
  - β the true phase difference, and
  - D the coherence.
- Challenge: vectorial data with channels of different nature
  - z, $z' \in \mathbb{C}$,
  - $R \in \mathbb{R}^+$,
  - $\beta \in [-\pi, \pi[$ (wrapped data),
  - $D \in [0, 1]$.

Extension to InSAR data

Statistical model of InSAR data

$$p(z, z'|\Sigma) = \frac{1}{\pi^2 det(\Sigma)} \times \exp\left[- \left(z^* z'^*\right) \Sigma^{-1} \begin{pmatrix} z \\ z' \end{pmatrix} \right]$$

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  - $\beta \in [-\pi, \pi[$ (wrapped data),
  - $D \in [0, 1]$.
- Suitable similarity criteria can be derived from this joint distribution.

Closed-form expressions for InSAR data

- The WMLE for InSAR data can be expressed as:

$$\begin{aligned} \hat{R}_s &= \frac{a}{N}, \\ \hat{\beta}_s &= -\arg x, \\ \hat{D}_s &= \frac{|x|}{a} \end{aligned} \qquad \text{with} \qquad \begin{aligned} a &= \sum_t w(s,t) \frac{|z_t|^2 + |z'_t|^2}{2}, \\ x &= \sum_t w(s,t)\, z_t z'^*_t, \\ N &= \sum_t w(s,t). \end{aligned}$$

- The similarity between noisy data is given by:

$$\sqrt{\frac{\mathcal{C}}{\mathcal{B}}}^3 \left(\frac{\mathcal{A}+\mathcal{B}}{\mathcal{A}} \sqrt{\frac{\mathcal{B}}{\mathcal{A}-\mathcal{B}}} - \arcsin \sqrt{\frac{\mathcal{B}}{\mathcal{A}}} \right)$$

with $\mathcal{A} = \left(|z_1|^2 + |z'_1|^2 + |z_2|^2 + |z'_2|^2 \right)^2$

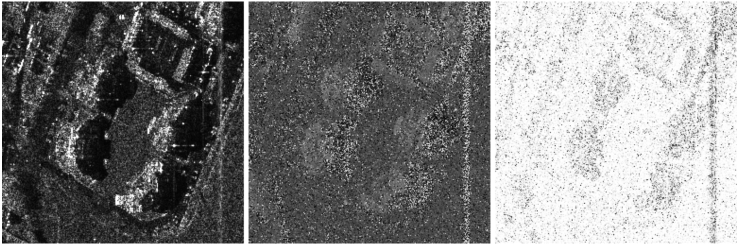
$\mathcal{B} = 4 \left| z_1 z'_1 + z_2 z'_2 \right|^2$

and $\mathcal{C} = \left| z_1 z'_1 z_2 z'_2 \right|$

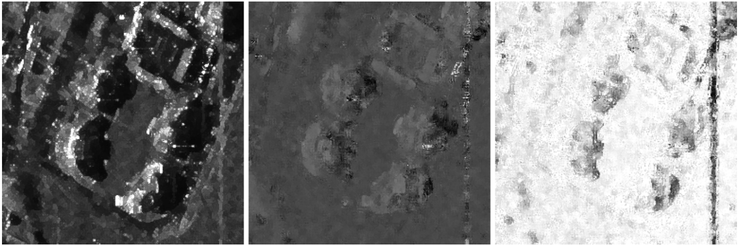
- The similarity between pre-filtered data is given by:

$$\frac{4}{\pi} \left[(1 - \hat{D}_1 \hat{D}_2 \cos(\hat{\beta}_1 - \hat{\beta}_2)) \left(\frac{\hat{R}_1}{\hat{R}_2(1-\hat{D}_2^2)} + \frac{\hat{R}_2}{\hat{R}_1(1-\hat{D}_1^2)} \right) - 2 \right]$$

NL-InSAR (Saint-Pol-sur-Mer, France, RAMSES ©DGA ©ONERA)



(a) InSAR-SLC



(b) Refined Lee [Lee et al., 2003]

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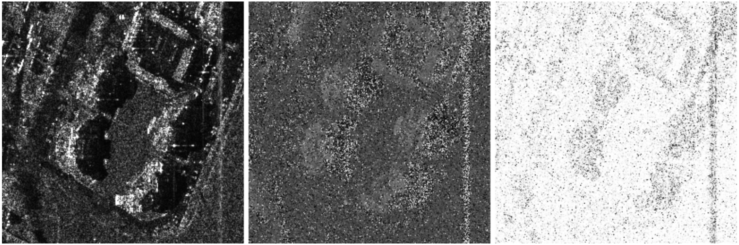

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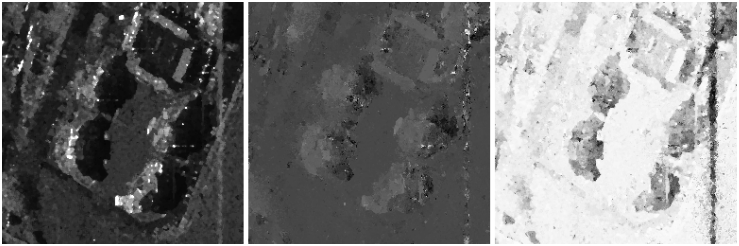

(b) IDAN [Vasile et al., 2006]

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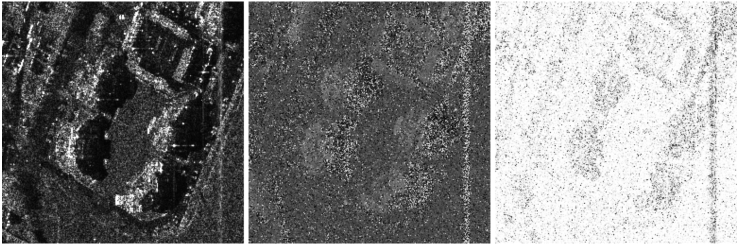

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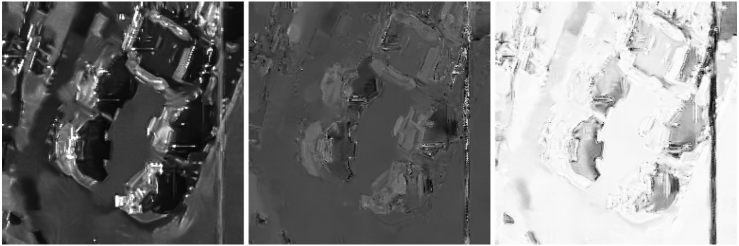
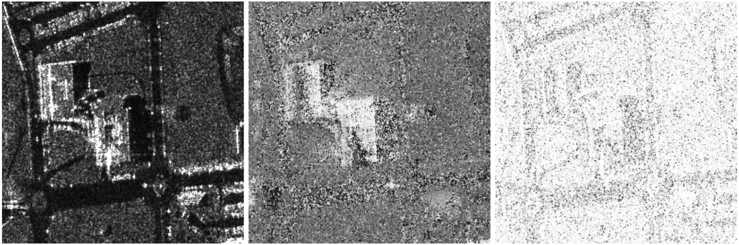


(b) **NL-InSAR**

Results of NL-InSAR (Toulouse, France, RAMSES ©DGA ©ONERA)



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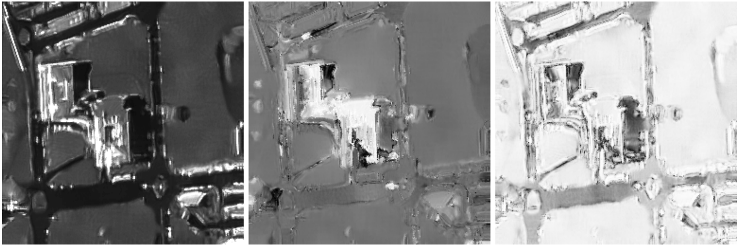
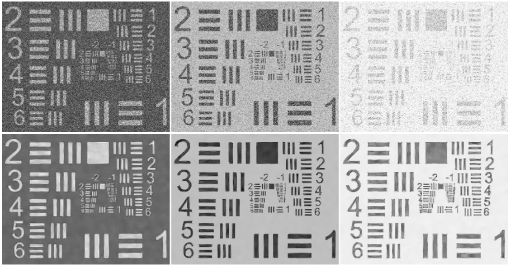


(b) **NL-InSAR**

Numerical results of NL-SAR and NL-InSAR on a resolution test-pattern

	SAR	InSAR		
Channel	R_{SAR}	R_{InSAR}	β	D
SLC	-4.42	-2.75	3.36	-1.19
Refined Lee [Lee et al., 2003]	5.47	6.23	9.12	2.03
WinSAR [Achim et al., 2003]	5.49	–	–	–
IDAN [Vasile et al., 2006]	–	5.00	7.88	0.33
NL-(In)SAR	**7.46**	**9.02**	**13.04**	**6.92**

SNR values of estimated SAR and InSAR images using different estimators



Results of NL-InSAR on the resolution test-pattern

Conclusion

- We proposed an efficient estimator of SAR and InSAR data based on non-local approaches,
- The idea is to search iteratively the most suitable pixels to combine,
- Similarity criteria:
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  - joint similarity between the pre-filtered data of surrounding patches.

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- More details in:

[Deledalle et al., 2010] Deledalle, C., Denis, L., and Tupin, F. (2010).
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website: `http://perso.telecom-paristech.fr/~deledall`
NL-SAR software available

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Iterative Weighted Maximum Likelihood Denoising with Probabilistic Patch-Based Weights.
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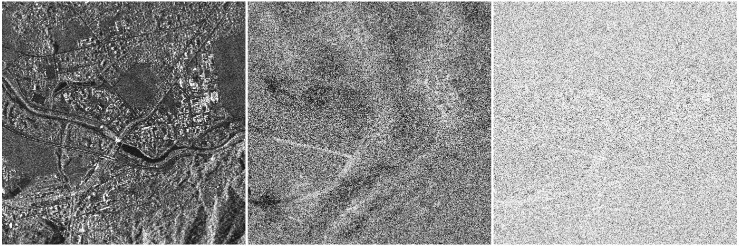
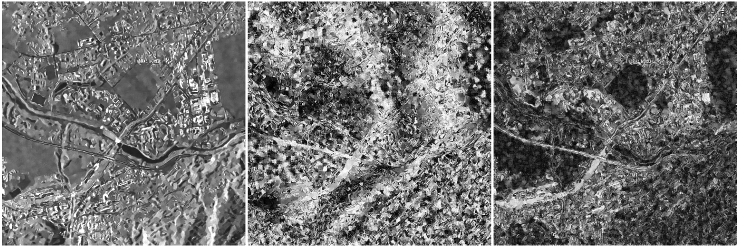
Results of NL-InSAR (Saint-Gervais, France, TerraSAR-X ©DLR)



(a) InSAR-SLC



(b) **NL-InSAR**

Distributions of SAR and InSAR data

- Distribution of SAR data, Rayleigh distribution:

$$p(A|R) = \frac{2A}{R} \exp\left(-\frac{A^2}{R}\right)$$

- Distribution of InSAR data [Goodman, 2006]:

$$p(z, z'|\Sigma) = \frac{1}{\pi^2 det(\Sigma)} \times \exp\left[-\left(z^* z'^*\right) \Sigma^{-1} \begin{pmatrix} z \\ z' \end{pmatrix}\right]$$

$$\Sigma = \mathbb{E}\left\{\begin{pmatrix} z \\ z' \end{pmatrix} \left(z^* z'^*\right)\right\} = \begin{pmatrix} R & \sqrt{RR'}De^{j\beta} \\ \sqrt{RR'}De^{-j\beta} & R' \end{pmatrix}$$

- which is equivalent to:

$$p(A, A', \Delta\phi|R, D, \beta) = \frac{2AA'}{\pi R^2(1 - D^2)} \times \exp\left(-\frac{A^2 + A'^2 - 2DAA' \cos(\Delta\phi - \beta)}{R(1 - D^2)}\right).$$

with $z = A^{j\phi}$, $z' = A^{j\phi'}$ and $\Delta\phi = \phi - \phi'$.

- WMLE for SAR data:

$$\hat{R}_s = \frac{\sum_t w(s,t) A_t^2}{\sum_t w(s,t)}$$

- WMLE for InSAR data:

$$\begin{aligned}
\hat{R}_s &= \frac{a}{N}, \\
\hat{\beta}_s &= -\arg x, \\
\hat{D}_s &= \frac{|x|}{a}
\end{aligned}$$

$$\text{with} \quad \begin{aligned}
a &= \sum_t w(s,t) \frac{|z_t|^2 + |z'_t|^2}{2}, \\
x &= \sum_t w(s,t)\ z_t {z'_t}^*, \\
N &= \sum_t w(s,t).
\end{aligned}$$

Closed-form expressions of similarities for SAR and InSAR data

Similarity between ...	SAR	InSAR
... noisy data	$\log\left(\frac{A_1}{A_2} + \frac{A_2}{A_1}\right)$	$-\log\left[\sqrt{\frac{\mathcal{C}}{\mathcal{B}}}^3 \left(\frac{\mathcal{A}+\mathcal{B}}{\mathcal{A}}\sqrt{\frac{\mathcal{B}}{\mathcal{A}-\mathcal{B}}} - \arcsin\sqrt{\frac{\mathcal{B}}{\mathcal{A}}}\right)\right]$ with $\mathcal{A} = \left(\lvert z_1\rvert^2 + \lvert z_1'\rvert^2 + \lvert z_2\rvert^2 + \lvert z_2'\rvert^2\right)^2$, $\mathcal{B} = 4\left\lvert z_1 z_1' + z_2 z_2'\right\rvert^2$, $\mathcal{C} = \left\lvert z_1 z_1' z_2 z_2'\right\rvert$
... pre-filtered data	$\frac{(\hat{R}_1 - \hat{R}_2)^2}{\hat{R}_1 \hat{R}_2}$	$\frac{4}{\pi}\left[(1 - \hat{D}_1\hat{D}_2\cos(\hat{\beta}_1 - \hat{\beta}_2))\left(\frac{\hat{R}_1}{\hat{R}_2(1-\hat{D}_2^2)} + \frac{\hat{R}_2}{\hat{R}_1(1-\hat{D}_1^2)}\right) - 2\right]$